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The Use and Abuse of Fibonacci Numbers and the Golden Section in Musicology Today

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A complex set of misconceptions about the use of the golden section and Fibonacci numbers in music has evolved in recent years.[1] The perpetuation of these misconceptions is guaranteed by a steady increase in the number of printed and electronic articles, books and dissertations, many of which bear the marks of academic integrity. The eminently respectable encyclopædia *Musik in Geschichte und Gegenwart* lists fifty items in the bibliography following the entry 'Goldener Schnitt'.[2] Disregarding the handful written 'with more vigour than discretion',[3] the majority are scholarly items from peer-reviewed publications. One of the most frequently-cited and best-argued articles is Allan Atlas' 'Gematria, Marriage Numbers and Golden sections in Dufay's 'Resvellies vous'' (*Acta Musicologica*, 1987). A sample paragraph from Atlas' text reads:

> Known to the architects of ancient Egypt and Babylonia, described by Euclid (as the extreme and mean ratio) and later praised for its magical and divine qualities by the likes of Leonardo da Vinci ('sectio aurea'), Luca Pacioli ('divina proportione') and Johannes Kepler ('sectio divina'), the Golden section of a given length may be arrived at by multiplying the whole by 0.618. And though the resulting proportion will always be expressed as an irrational fraction, its equivalent in rational numbers can be approximated by means of the number

1 I am grateful to the Trustees of the Hinrichsen Foundation for funding the research of the original paper given at the Oxford Bach Symposium in 2002. I am also grateful to the maths historians Professor Ivor Grattan-Guinness (Middlesex University) and Professor Roger Herz-Fischler (Carleton University) for their stimulating correspondence and conversations on this subject.

2 *Die Musik in Geschichte und Gegenwart*, rev. ed. (1998) s.v. 'Goldener Schnitt'.

3 A memorable phrase coined by the late Malcolm Boyd in *Bach: The Master Musicians* (London, Melbourne: Dent, 1983), p. 223, to describe another well-known numerical technique that has tended to attract musicology's lunatic fringe.

> sequence devised by the 13th century mathematician Leonardo da Pisa in his *Liber abbaci* of 1202, which more and more closely approximates the Golden section as it approaches its higher terms.[4]

This sounds plausible and appears to be factually correct, but on closer study it implies much that is incorrect. For example, the golden section was indeed praised for its magical and divine qualities, but da Vinci and Pacioli found it purely through geometry, by means of compasses, and not through multiplication by a decimal, nor even through a numerical sequence. Kepler, on the other hand, knew that the number sequence closely approximated Euclid's ratio, but he did not know that Fibonacci had written about it in 1202. Atlas implies that Fibonacci was aware that his rabbit sequence was a numerical approximation of Euclid's ratio. Working in good, albeit misguided, faith that his historical understanding was correct, Atlas took his argument to its logical conclusion, and stated that Dufay used a golden section consciously to encapsulate the symbolic meaning of 'Resvellies vous' in the 271st minim.

> Dufay's 'Resvellies vous' is 438 minims long; the point of division between the longer and shorter parts of the Golden section then is 270.684, or more practically, the 271st minim, and thus falls on the downbeat of the 46th breve. The significance seems clear: Dufay marks the structural point of the piece as a whole with the melodic and articulatory high point of the very phrase that contains in a nutshell the symbolic meaning of the work.[5]

Atlas' method typifies the problems associated with the majority of musicological writing on the golden section. His methodology was flawed; he failed to appreciate a distinction between the composer's conscious compositional effort and the properties of the composition. The discovery of a golden section in a composition is no proof that the composer planned it. Atlas' historical knowledge was also limited. He did not know that Fibonacci's papers were first rediscovered in the nineteenth century, nor did he know that Fibonacci himself was unaware that his rabbit sequence was an approximate expression of the so-called 'golden section'. Putting together the facts that the so-called golden section[6] had been praised for its magical and divine qualities in 1509, and that Fibonacci had written his sequence in 1202, led Atlas to believe that Dufay, composing in 1423, used Fibonacci's sequence to emulate the 'magical and divine' ratio. In reality, Dufay could not consciously have incorporated this structural division into his work at the 271st minim without the numbers of Fibonacci's sequence to help him.

I am writing this paper in the hope that it will contribute to halting the perpetuation of such misconceptions, and even stimulate new and historically-accurate studies of this aesthetically-intriguing phenomenon.

4 Allan Atlas, 'Gematria, Marriage Numbers and Golden Sections' in Dufay's 'Resvellies vous', *Acta Musicologica* 59 (1987), p. 122.

5 Atlas, *Acta Musicologica,* 59, pp. 122-23.

6 The phrase golden section, or 'goldener Schnitt' was first formulated in 1835, see section I iii) b) below.

I Historical evidence from maths history

i) History and transmission of Euclid's ratio

In about 300 BCE, Euclid described a ratio which he termed Extreme and Mean Ratio.[7] A line is said to be divided in Extreme and Mean Ratio, when the line AB is divided into two unequal parts so that the ratio of the whole line (AB) to the larger part (AC) is the same as the larger part (AC) to the smaller part (CB).

Division in Extreme and Mean Ratio (DEMR - Euclid)

A______________________________C_________________B

AB:AC=AC:CB

Euclid's ratio is known by many names including the 'golden section', 'divine proportion', and 'Division in Extreme and Mean Ratio (DEMR)'. The terms 'divine proportion' and 'golden section' to describe Euclid's DEMR were coined in the sixteenth and nineteenth centuries respectively. Following the practice of mathematicians, I shall use the abbreviation DEMR when referring to Euclid's ratio, to avoid using 'golden section' with its many aesthetic allusions.[8]

DEMR is an ancient Greek division, described several times in Euclid's *Elements:* a) in the construction of a rectangle and a square (Book II prop. 11, and Book VI, prop. 30); b) in the construction of a regular pentagon (Book IV prop.11and Book XIII prop. 8), and c) in the construction of the icosahedron (Book XIII prop. 16) and the dodecahedron (Book XIII prop. 17). DEMR is formed by lines and compasses. Euclid never mentions the size of the angle. He has no numerical expression for DEMR. General rational numbers do not occur in his *Elements.* For Euclid, numbers and magnitudes were different kinds of quantity; arithmetic dealt with the discrete, and geometry with the continuous.[9] Euclid never uses the term 'golden' or 'section' to describe DEMR. Thomas Heath tried to argue that Proclus' earlier use of the term 'section' was a reference to DEMR,[10] but this has since been soundly refuted.[11]

7 Roger Herz-Fischler, *A Mathematical History of Division in Extreme and Mean Ratio* (Waterloo, Ontario: Wilfrid Laurier University Press, 1987), repr. as *A Mathematical History of the Golden Number* (New York: Dover, 1999) pp. 164-65 in both editions.

8 *ibid*. Without Roger Herz-Fischler's exhaustive study I could not have begun to understand or unravel the complexities of this subject and its implications for musicology.

9 Ivor Grattan-Guinness, *The Fontana History of the Mathematical Sciences* (London: Fontana, 1997), pp. 60-61.

10 Sir Thomas L. Heath (trans.), *Euclid: the Thirteen Books of the Elements* (New York: Dover, 1956), 2nd edn, pp. 98-99.

11 Herz-Fischler, *A Mathematical History*, pp. 96-99 and p. 167.

Even though the transmission of Euclid's ratio is long and interesting in mathematical terms, for our purposes the first milestone is Luca Pacioli's work. This is not because Pacioli added anything special to the mathematics, but because he introduced the emotive adjective 'divine' into the discussion.

Pacioli's *De Divina Proportione,* or *On the Divine proportion* is the title for the first work and also the collective title for all three works published together in 1509; *On the Divine proportion*, *Architectural Treatise* and *Libellus*. In *On the Divine proportion* Pacioli describes the divine proportion as

> essential, singular, ineffable, admirable, unnamable, inestimable, surpassing all others, most excellent, almost incomprehensible, most worthy . . . this proportion of ours cannot ever be designated through intelligible number, nor can it be expressed through any rational quantity, but always remains occult and secret, and is called irrational by the mathematicians.[12]

Pacioli could not express the divine proportion in integers, however marvellous he thought the ratio was.

The composer Dufay lived between c. 1400 and 1474, and was working a good fifty years before Pacioli's book was published. It would have had to have been a quirk of fate for Dufay to have used rational numbers to express DEMR. Since mathematicians praising the ratio for its divine nature did not have such numbers, is it likely that Dufay did? And, given this historical situation, how valid are all those musicological claims about the *conscious* incorporation of golden sections in music composed pre-1509?

Pacioli's second volume *Architectural Treatise* is based on Vitruvius' work and discusses proportions in architecture, the human body and lettering. He classifies the proportions into thirty-nine subdivisions, and illustrates the formation of letters, architecture, and the composition of the human form. The proportions are all simple, and use rational numbers. As this second book is bound in with the mathematical treatise *Divine proportion*, it has been falsely assumed that Pacioli was applying DEMR to architecture. Herz-Fischler traces this error back to 1799,[13] when, in the second edition of his *Histoire des mathématiques*, Montucla described Pacioli's work:

> A large part of this work consists of plates representing the application of the divine proportion in architecture, the formation of capital letters, which are of such good taste that I suspect they are taken from ancient monuments.[14]

Montucla's false assessment has been repeated countless times, and is deeply engrained in the popular myth of the golden section. Pacioli did not apply DEMR to lettering, architecture or the human form.

The third volume in Pacioli's *On the Divine proportion* has the title *Libellus.* It is a line-by-line Italian translation of Piero della Francesca's Latin study of the five

12 *ibid*. p. 172.

13 *ibid*. p. 150.

14 J. F. Montucla, *Histoire des mathématiques* (Paris, 1799), p. 551.

Platonic Solids 'De quinque corporibus' (1478).[15] Neither Piero della Francesca nor Pacioli use Fibonacci numbers to discuss the construction of the solids. And this is where Leonardo da Vinci enters the discussion. Leonardo is linked to Pacioli because he illustrated Pacioli's *Divine proportion.* Leonardo drew the Platonic Solids. Leonardo knew Euclid's ratio. He used compasses and could easily reproduce DEMR,[16] but he neither knew nor used rational numbers to express it.

Fifty years ago, after an exhaustive study of art and architectural treatises of the period, the art historian Rudolf Wittkower came to the following conclusion:

> To my mind, Renaissance art theorists omitted discussing the golden section because its irrational properties could not be reconciled with Alberti's 'reliable and definite annotation of dimensions' (*De Pictura*). Moreover the golden section was only of secondary importance in practice. In the most important group of Renaissance studies in proportion, those by Leonardo, we never find a deliberate use of irrational magnitudes.[17]

It is a lamentable state of affairs that musicology failed to take note of the writings of an eminent colleague in such a closely-related discipline.

To summarise the situation in 1509:

a) DEMR was well known. It was formed by using lines, angles and compasses. Therefore it could be reproduced in paintings, if the artist used compasses, and in architecture and building construction, when the builders used angles and plumb lines.
b) There were still no rational numbers to express it.
c) In this period rational numbers were all-important.
d) The theory of proportions dominated music and art treatises of the time.
e) Additive sequences were of no interest to mathematicians.
f) Without a numerical expression of DEMR, composers could not have consciously used it to construct their compositions.

ii) History and transmission of the so-called 'Fibonacci sequence'

Leonardo da Pisa, otherwise known as Fibonacci, described an additive sequence in *Liber Abbaci* (1202), using an apocryphal story about rabbit breeding as an illustration.

> How many pairs of rabbits can be bred from one pair in one year? A man has one pair of rabbits at a certain place entirely surrounded by a wall. We wish to know how many pairs can be bred from it in one year, if the nature of these rabbits is such that they breed every month one

15 The five Platonic Solids are the regular tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron.

16 Martin Kemp (ed.), *Leonardo on Painting* (New Haven, London: Yale University Press, 1989), pp. 119-23.

17 Rudolf Wittkower, *Architectural Principles in the Age of Humanism*, 5th edn (Chichester: Academy Editions, 1998), Appendix IV p. 153.

> other pair, and begin to breed in the second month after their birth. Let the first pair breed a pair in the first month, then duplicate it and there will be 2 pairs in a month. From these pairs one, namely the first breeds a pair on the second month, and thus there are 3 pairs in the second month. From these in one month two will become pregnant, so that in the third month 2 pairs of rabbits will be born. Thus there are 5 pairs in this month. From these in the same month 3 will be pregnant, so that in the fourth month there will be 8 pairs. From these pairs 5 will breed 5 other pairs, which, added to the 8 pairs gives 13 pairs in the fifth month, from which 5 pairs (which were bred in that same month) will not conceive in that month, but the other 8 will be pregnant. Thus there will be 21 pairs in the sixth month. . . Finally there will be 377. And this number of pairs has been born from the first-mentioned pair at the given place in one year. You can see in the margin how we have done this, namely by combining the first number with the second, hence 1 and 2, and the second with the third, and the third with the fourth. At last we combine the 10th with the 11th hence 144 with 233 and we have the sum of the above-mentioned rabbits, namely 377, and in this way you can do it for the case of infinite numbers of months.[18]

The rabbit sequence became known as the Fibonacci sequence much later. It was not until 1857 that Fibonacci's *Liber Abbaci* was published by Prince Baldassarre Boncompagni in two-volumes.[19] Targioni Tozzetti had apparently discovered the original manuscript in the Italian library in 1745, but as the knowledge raised little more than a footnote, Fibonacci's writings lay undisturbed for a further hundred years.[20] The rabbit problem stands by itself in *Liber Abbaci,* situated between a problem dealing with perfect numbers (6, 28, 496 etc.)[21] and a solution for a system of four linear equations with four unknowns.[22] At no point does Fibonacci make a connection between the sequence and Euclid's ratio, and it appears he was unaware of the properties of his sequence. And so, it follows that, even if a manuscript copy of Fibonacci's *Liber Abbaci* had somehow fallen into Dufay's hands, and he had read the relevant entry, Dufay would not have recognised the rabbit sequence as an expression of Euclid's ratio.

18 D. J. Struik (ed.), *A Source Book in Mathematics, 1200-1800* (Cambridge, Mass.: Harvard University Press, 1969), p. 2.

19 Leonardo da Pisa, 'Scritti di Leonardo Pisano matematico del secolo decimoterzo' in Baldassarre Boncompagni (ed.), *Liber Abbaci* Vols. 1-2 (Rome: 1857, 1862).

20 John Leslie, *The Philosophy of Arithmetic* (Edinburgh, 1817), footnote on p. 226.

21 A perfect number is one whose divisors add up exactly to the number itself. The number 6 has the divisors 1, 2, 3, and 1+2+3=6. 28 is the next perfect number because 1+2+4+7+14=28. 496 is the third, 8128 the fourth.

22 Leslie, *The Philosophy*, p. 226, and Leonardo da Pisa 'Scritti di Leonardo Pisano', Vol. I pp. 283-84.

iii) History and transmission of the sequence expressing DEMR

a) Simon Jacob and Johannes Kepler

The numerical sequence had been known long before it took the name Fibonacci. Not much arithmetical skill is required to reproduce an additive sequence, but great skill is required to understand the properties, applications and implications of such a sequence. Throughout the history of mathematics the sequence occasionally occurs to express DEMR. Early appearances seem to be unrelated to each other, as if the individual mathematician had discovered the solution himself.

Searching for the earliest link between Euclid's ratio and the sequence, Leonard Curchin and Roger Herz-Fischler found an undated handwritten annotation in Pacioli's 1509 edition of Euclid's *Elements*.[23] The annotation in a sixteenth-century hand provides a numerical solution for Euclid's Proposition 11 from Book 2, giving the numbers 233, 89 and 144 along the sides divided in EMR. Where did this knowledge come from, the authors asked, since it did not come from any of the great mathematicians, such as Piero della Francesca nor Bombelli, living between 1526 and 1572?

Until recently it was thought that Johannas Kepler was the first person to print a numerical expression of DEMR, but in 1995 Peter Schreiber changed this when he published a note in *Historia Mathematica*.[24] Schreiber announced that Simon Jacob, who died in 1564, had published a numerical solution for DEMR. In the margin of the page discussing the Euclidean algorithm from the second proposition of Book 7, Jacob wrote the first twenty-eight terms of the 'Fibonacci sequence', and stated:

> In following this sequence one comes nearer and nearer to that proportion described in the 11th proposition of the 2nd book and the 30th of the 6th book of Euclid, and though one comes nearer and nearer to this proportion it is impossible to reach or to overcome it.[25]

Because the remark linking DEMR to the sequence bears no relation to Jacob's stated aim of investigating the reduction of fractions, Schreiber thinks that Jacob either copied it from another source, or had discovered it himself, and 'could not resist reproducing his interesting find'.[26]

[23] Leonard Curchin and Roger Herz-Fischler, 'De Quand date le premier rapprochement entre la suite de Fibonacci et la division en extrême et moyenne raison?', *Centaurus* 28 (1985), pp. 129-38.

[24] Peter Schreiber, 'A Supplement to J. Shallit's Paper "Origins of the Analysis of the Euclidena Algorithm"', *Historia Mathematica* 22 (1995), pp. 422-24.

[25] I have adapted Schreiber's translation. The original reads: 'je weiter mann solche ordnung / je näher mann zu der Proportion kompt / Davon beim Euclide die 11 Proposi. des 2 / unnd 30 des 6 Buchs handeln / unnd wiewol man immer ihe näher kompt / mag doch nimmermehr dieselb erreicht auch nicht übertretten werden.'

[26] In 1985 I read a copy of this extremely rare book, the 1565 edition, A:Wn-72.H.32, but missed this significant link between the sequence and Euclid, as I was looking for another use: see Ruth Tatlow, *Bach and the Riddle of the Number Alphabet* (Cambridge: Cambridge Univesity Press, 1991), pp. 52 and 133.

In a private letter to Professor Joachim Tanckius dated 1618, Johannes Kepler explained how he found the numerical expression of DEMR:

> Now the divine proportion cannot, however be perfectly expressed in numbers: it can nevertheless be expressed in such a way that through an infinite process we come closer and closer to it, and in delineating the square we are never more than a unity away.[27]

In 1611 Kepler published his discovery in *Six-Cornered Snowflake*:

> Of the two regular solids, the dodecahedron and the icosahedron, the former is made up precisely of pentagons, the latter of triangles, but triangles that meet five at a point. Both of these solids, and indeed the structure of the pentagon itself, cannot be formed without this proportion that the geometers of today call divine . . . It is impossible to provide a perfect example in round numbers. However, the further we advance from the number one, the more perfect the example becomes. Let the smallest numbers be 1 and 1. . . Add them, and the sum will be 2; add to this the greater of the 1s, result 3; add 2 to this, and get 5; add 3, get 8; 5 to8, 13; 8 to 13, 21. As 5 is to 8, so 8 is to 13 approximately, and as 8 to 13, so 13 is to 21, approximately.[28]

Recurring series became a fashionable area of study for algebraists in the late seventeenth and early eighteenth centuries,[29] but in 1611 Fibonacci was still hidden in the Roman library, and the mathematical discussion was unrelated to music.

b) Golden numberism

The practice of searching for golden sections, or golden numberism, did not begin until the 1830s. The interest was initially confined to Germany, but gradually spread to the rest of Europe. The major events progress can be charted in the following chronology.

1835: Martin Ohm changed the workaday term 'stetige Proportion' to 'Goldener Schnitt' ('golden section') in a revised edition of his maths text book, thus giving Euclid's EMR an attractive name.[30]

1854: Adolph Zeising searched for and found the numerical sequence and golden sections in natural phenomena and art, for example in the sunflower, in the ratios of the human body and in classical sculpture.[31]

1855: Röber claimed that the golden section was used to design almost all the Egyptian pyramids.[32]

27 Herz-Fischler, *A Mathematical History*, p. 161.

28 *ibid.*

29 Leonard Eugene Dickson, *A History of the Theory of Numbers*, Vol. 1 *Divisibility and Primality* (Washington: Carnegie Institute of Washington, 1919), pp. 393-411.

30 *ibid.* p. 1582.

31 A. Zeising, *Neue Lehre von den Proportionen des menschlichen Körpers* (Leipzig, 1854). See http://herz-fischler.ca/.

32 Roger Herz-Fischler, 'The Golden Number, and Division in Extreme and Mean Ratio', in I. Grattan-Guinness (ed.) *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*, Vol. 2 (London: Routledge, 1994), p. 1580.

1857: Fibonacci's manuscripts were published in Rome in two volumes by Baldasarre Boncompagni.[33]
1878: the French mathematician Edouard Lucas (1842-1891) coined the phrase 'the Fibonacci sequence'.[34]

Golden numberism was a late nineteenth century-German fashion that spread rapidly after 1910 to other countries like 'a sudden and devastating disease which has shown no signs of stopping'.[35] It was in response to this movement that composers from the late nineteenth century onwards began to experiment with the aesthetic properties of the golden section (the correct terminology after 1835) through the use of the Fibonacci sequence (the correct terminology after 1878). But what about earlier composers? Were they interested in investing their works with the golden section, and if so, how did they do it?

II Evidence from music history

i) Boethius (480-524 CE) and Nicomachus (fl. 100 CE)

Rumour has it that Boethius described the Fibonacci sequence in *De Institutione Arithmetica*, which is largely a translation of Nicomachus' *Arithmetike Eisagoge* (*Introduction to Arithmetic)*. Allan Atlas mentions Nicomachus' use of the sequence, and cites Powell as his source. Newman Powell, writing in 1979 in the *Journal of Music Theory,* claims that Nicomachus' tenth proportion uses the Fibonacci sequence. The thoroughgoing mathematicians and maths historians Dickson (1919) and Herz-Fischler (1987, rev. 1998) do not mention Nicomachus' tenth proportion in the context of the Fibonacci sequence. Powell, a musicologist, however, writes:

> While the names for the Fibonacci and Lucas series were not established until the 19th and 20th centuries, the series themselves were known throughout the Middle Ages in the form of the Neo-Pythagoreans' 'tenth proportion', which states the additive process involved in any Fibonacci sequence.[36]

The implications of this statement are enormous. If Nichomachus' tenth proportion describes the Fibonacci sequence, then it follows that Boethius knew it. As Boethius' writings are the foundation stone of western music theory, the sequence may consequently lie deep at the heart of western music.[37] Nicomachus

33 Leslie, *The Philosophy*, footnote on p. 226.
34 Grattan-Guinness, *The Fontana History*, p. 751.
35 Cook quoted in Herz-Fischler, 'The Golden Number', p. 1582.
36 Newman W. Powell, 'Fibonacci and the Gold Mean: Rabbits, Rumbas and Rondeaux', *Journal of Music Theory* 23 (1979), 230.
37 Michael Masi, *Boethian Number Theory: A Translation of* De Institutione Arithmetica (Amsterdam: Rodopi, 1983); Nicomachus' 10th proportion appears on p. 170. On p. 33 note 24 Masi states that 'a description of the Golden Mean does not occur in *De Institutione Arithmetica,* but may be found briefly stated in *De Geometrica*. Friedlein p. 386.' The description in *De Geometrica* contains no numbers for the proportion – it is purely a geometric description.

was writing in the first century CE. He discusses proportions in Book Two of *Eisagoge arithmetica*.

> There are six proportions commonly spoken of among previous writers, the prototypes having lasted from the times of Pythagoras down to Aristotle and Plato, and the three others, opposites of the former, coming into use among the commentators and sectarians who succeeded these men. But certain men have devised in addition, by shifting the terms and differences of the former, four more which do not much appear in the writings of the ancients, but have been sparingly touched upon as an over-nice detail. These, however, we must run over in the following fashion, lest we seem ignorant.[38]

Nicomachus' tenth proportion is among the last four, which he included for the sake of completion, 'lest he seem ignorant'. He continues:

> The tenth proportion in the full list is seen when, among three terms, as the mean is to the lesser, so the difference of the extremes is to the difference of the greater terms, as 3, 5, 8 for it is the superbipartient ratio in each pair.

In Nicomachus' example, terms 3 and 8 are the extremes and 5 is the mean. As the mean (5) is to the lesser (3) i.e. 5:3, so the difference of the extremes (8-3, so 5) is to the difference of the greater terms (8-5, so 3), i.e. 5:3=5:3. The ratio 5:3 happens to be called a superbipartient ratio, but the term is relevant only to this one example. Nicomachus could easily have chosen three other terms. As Powell rightly says, 'the statement of this 10th proportion seems inconsequential, even trivial, for it occurs any time two numbers are added together to make a third'.[39] Note that in his description of the tenth proportion, Nicomachus makes no mention of DEMR. The tenth is just one in a list of all known proportions. Powell continues:

> Another condition that would make the 'tenth proportion' more significant occurs when the given proportion is far enough into its own distinctive series that the ratios produced approach the ratios found in the Golden section, that 'magic' ratio that held such a fascination for mathematicians and artists down through the ages.

Powell is warming to his theme, and instead of keeping strictly to logic and facts, he begins to overstep the limits of Nicomachus' text. Powell was reminded of the Fibonacci sequence when he saw Nicomachus' example, the ratio 3:5:8.
He continues:

> It is only when one recognises the Fibonacci process (or more properly, now, the 'Golden Sequence process') revealed by the expansion of the 'tenth proportion' into a continuous proportion that it becomes consequential.[40]

Powell has moved away from the primary source. He does not actually state that Nicomachus is describing the Fibonacci sequence, but works by suggestion. He

38 Nicomachus of Gerasa, *Introduction to Arithmetic*, trans. M. L. D'Ooge with studies in Greek arithmetic by F. E. Robbins and L. C. Karpinski (New York: Macmillan, 1926), pp. 285-86.

39 Powell, 'Fibonacci and the Gold Mean', p. 230.

40 *ibid*. p. 239.

introduces the divine proportion, while the reader is still wondering whether Nicomachus' superbipartient example was anything to do with the Fibonacci sequence. This is extremely disappointing scholarship. Careful reading of Nicomachus shows that he was describing a proportion unrelated to Euclid's DEMR, its significance or to any additive sequence.

I was further disturbed by Powell's scholarship when I read the citation in context. Just thirty-three words down the page,[41] in the paragraph immediately following the tenth proportion, Nicomachus writes:

> It remains for me to discuss briefly the most perfect proportion, that which is three-dimensional and embraces them all, and which is most useful for all progress in music and in the theory of the nature of the universe. This alone would properly and truly be called harmony rather than the others.[42]

What devastating proof had the proportion 5:8:13 been the most perfect, and most useful for all progress in music. But it was not. Nicomachus' three-dimensional perfect proportion most useful for all progress in music is 6:8:9:12, with no mention of a proportion or sequence tending to DEMR. I find it perplexing that in an article purporting to represent Nicomachus' view of music should fail to mention the only proportion which Nicomachus relates to music. Powell continues:

> Whether composers actually worked with geometric drawings and figures as a guide to determining proportions in durations we cannot say, but when we find identical additive processes in the geometry of the Golden Sequence and in Nicomachus' tenth proportion, it seems possible that these processes might have been consciously exploited by composers of the 14th and 15th centuries. Indeed one is tempted to speculate that the 'tenth proportion' was introduced by the Neo-Pythagoreans in order to provide an arithmetic approach to the Golden Sequence process.[43]

And tempted he was. Through numerical sequences, complex drawings and figures he demonstrates the appearance of the golden section in the music of Dufay and Machaut. Powell's article has become a standard reference source. The myth of Fibonacci numbers appearing in Nicomachus, Boethius and music of the fifteenth century is currently embedded deeply in the substrata of medieval and Renaissance musicology.

ii) Salinas, Mersenne and Descartes

Although fairly clear principles for the non-use of the 'Fibonacci sequence' and the golden section in music can be drawn from the history of mathematics before c. 1600, there is a grey area for music composed in the period between 1600 and

41 Thirty-three words in the English translation. I have not counted the number of words in the original Greek.

42 Nicomachus, *Introduction to Arithmetic*, pp. 284-85.

43 Powell, 'Fibonacci and the Gold Mean', p. 239.

the rise of golden numberism in the 1840s. During this time several mathematicians worked on formulae that demonstrate an association between the additive sequence and DEMR. The association is called the Binet formula, although Binet himself did not formulate it until 1843.

There were also several isolated experiments involving the positioning of the fretting of the lute according to DEMR. This began with Salinas in 1577 and was corrected by Mersenne in 1636. Mersenne gives clear geometrical instructions as to how to cut a line in mean and extreme ratio.[44] Note that there is no mention of rational figures for these calculations. The divisions are made with compasses.

Mersenne worked in close collaboration with René Descartes, whose *Compendium Musicae* was published posthumously in 1650. It appeared three years later in an English translation, accompanied by an anonymous critical commentary, presumed to be by Brouncker.[45] The English translation and commentary contains a discussion of tuning systems using DEMR, and attempts to describe them in logarithms.[46] The discussion is purely theoretical and the anonymous author himself admits that it has no practical application: 'But this exactnesse is not requisite, since the Sense of Hearing is not so perfect, as to confine the Consonantes to so precise a Measure': a statement that would surely have frustrated contemporary musicians.[47] However, the examples are significant as they link DEMR with the practical trades of musical instrument building and tuning.

iii) J. S. Bach and J. G. Walther

Bach was composing between around 1700 and 1750, a period in which it was theoretically possible for a composer to use the additive arithmetical sequence. Bach was interested in tuning systems and organ building, and he would certainly have read Mersenne, and seen the theoretical possibility of finding the point of DEMR on an instrument he was building. But we must remember that he could equally well have done the same in school geometry lessons. More significant for composition is Kepler's formulation. Kepler's works were not widely available in the late seventeenth century. In 1718, however, Michael Gottlieb Hansch decided to publish Kepler's complete works in Leipzig. In the event, it seems that only the first of the planned twenty-two volumes was printed. It is a significant event nonetheless, as it illustrates a concrete theoretical

44 Marin Mersenne, *Harmonie Universelle. The Books on Instruments*, trans. Roger E. Chapman (The Hague: Martinus Nijhoff, 1957), proposition XVII, 284 and proposition VII, pp. 108-09; Marin Mersenne, *Harmonie Universelle contenant La Théorie et la Pratique de la Musique Paris, 1636* (Paris, 1963), pp. 224-25.

45 Lord Brouncker was a renowned free-thinker in his day. He turned his great intellect to ship design, as well as to mathematics and its application to music, and later became President of the Royal Society.

46 *Renatus Descartes Excellent Compendium of Musick with Necessary and Judicious Animadversions Thereupon* (London, 1653), pp. 66-68 and 84-94. I am grateful to Roger Herz-Fischler for alerting me to this example. See also Herz-Fischler, *A Mathematical History*, 1998, p. 178 [162, c.1].

47 *Renatus Descartes*, p. 89.

possibility, however slim, that Bach might have read Kepler's letter to Tanckius and seen the numerical expression for DEMR.[48]

Knowing and applying a fact are two different matters. I have shown that Bach and his contemporaries may have known the numerical sequence to express DEMR, but this does not mean that they used it in their compositions. In fact, no-one has yet found a single musical treatise from Bach's time or earlier that encourages a composer to emulate the divine proportion in composition. Bach's reading, however, went far beyond music treatises, and it is important to look beyond the parameters of his possible musical knowledge to see if there was an interest in this proportion in the society in which he lived. Zedler's sixty-eight-volume *Großes Vollständiges Universal Lexicon aller Wissenschaften und Künste* (the Complete Universal Encyclopædia of all Sciences and Arts) was published in Leipzig and Halle between 1732 and 1754, and represents the state of world knowledge viewed from a Leipzig perspective at that time. If the golden section, Fibonacci numbers or divine proportion had been fashionable or important they would receive full coverage in this colossal work.[49] After many failed attempts under obvious word searches, the entry 'Section Divina' (Divine section) produced a definition:

> SECTIO DIVINA, or DEMR, is a line AB divided at point C, whereby the ratio of the whole line AB is related to the larger part BC in the same ratio as the larger part BC is related to the smaller part AC. Euclid teaches how to divide a line in such a way in the Elements Bk II proposition 11, as does Wolff in Element. Analys. Finit. §220. Euclid found this division very useful for many complex demonstrations; (as they contain the aforementioned division in their wide-ranging applications) and de la Hire similarly makes extensive use of it in his Sectionibus conicis.

I have searched in vain for further definitions containing the ratio. There is no entry under the words: 'Sectio Aurea', 'Divini', 'Divini Numeri', 'Divini Proportioni', 'Numeri Divini' or 'Stetige Proportion'. The entry 'Zahl' (Güldene), Latin 'Numerus Aureus' looks promising, but this particular golden number concerns the lunar calendar and has nothing to do with DEMR. 'Continuum' discusses Wolff's Metaphysics. 'Divisio' does not mention DEMR. 'Progression' includes a description of the arithmetic progression, 3.5.7.9.11, but there is no mention of Fibonacci's additive sequence. 'Unendliche Reyhe', meaning 'infinite series', describes an early form of calculus and names Mercator, Wallis, Leibnitz, Newton and Brouncker, but nothing about our sequence or ratio. 'Proportio' covers fifteen and a half columns of text with ninety-four separate entries, none of which mentions DEMR or Nicomachus' tenth proportion. The absence of more references is ominous. Were the sequence and DEMR viewed in Leipzig as 'so

48 Michael Gottlieb Hansch, *Epistolae ad Joannem Kepplerum Mathematicum Caesareum* (Frankfort on the Main, 1718), pp. 405-18. On p. 171 of *A Mathematical History*, Herz-Fischler erroneously states that the Tanckius letter was not published until 1858-1871 in a complete edition by C. Frisch.

49 http://www.zedler-lexikon.de.

magical, so meaningful for mathematicians throughout the ages', it would have warranted more than one entry in this massive reference work. Zedler's sole definition is a technical description in geometrical terms, with neither a numerical equivalent for the ratio, nor a single word about its aesthetic properties.

There is a similar absence of information for Fibonacci or 'Da Pisa', although there is an entry s.v. 'Leonardus Pisanus', which reads:

> Leonardus Pisanus, a Mathematician who wrote about Algebra not long after 1400. Vossius, in his *Scientifica Arithmetica* p. 51 paragraph 8 notes that Lucas Paciolus a Minorite, acquired much of his work and made it famous in his Italian works published in Venice in 1494 and 1523.

There is no mention of the rabbits, the 'Sectio divina' or of a numerical sequence approximating DEMR. The two-hundred-year error in Fibonacci's dates is quite common.[50] The entry for Pacioli, 'Paciolus, Lucas (Luca de Burgo) Franciscan', is similarly silent on the value of his work to artists. It lists six publications, the first of which is '*Compendium de divina proportione.*Venice 1509'. Kepler's entry is more rewarding, with one-and-a-half columns of biography and a list of publications. The short commentary reads:

> Many of his thoughts were strikingly different, even peculiar . . . He was an amazing mathematician . . . Michael Gottlieb Hansch, who intended to publish his complete works in 22 volumes, made a start by publishing the letters and a biography in 1718.

The closest musical equivalent of Zedler's reference work is the *Musicalische Lexicon* (Leipzig, 1732) written by Bach's relative and exact contemporary Johann Gottfried Walther (1684-1748). Unfortunately it is published in just one slim volume, a fraction of the size of Zedler. The organisation of the multiple entry on 'Proportion' is similar to that in Zedler, and similarly fails to list 'Divine proportion'. There is no entry for DEMR, Sectio divina, or the sequence. In the short entry 'Kepler', Walther writes: 'Among very many works, he wrote one consisting of five books called *Harmonices Mundi*, which was published in Linz in 1619', but he makes no reference to snowflakes or to scientific correspondence with Tanckius, which may indicate that this aspect of Kepler's work was little known, or of little significance to the musician.

However, Walther's musical dictionary yields one surprising entry, 'Numerus perfectus'. In it he defines perfect numbers,[51] refers the reader first to Matthei's *On the Division of the Modes*, and then to the thirty-sixth proposition of the ninth book of Euclid. His entry is a word-for-word quotation from Matthei.[52] It is

50 Leslie, *The Philosophy*, p. 226 repeats it and explains how Fibonacci's works were discovered in the mid-1700s and then lost again in libraries as, in the meantime, history changed his name and his works were still catalogued under Bigollone or Bighelone rather than Fibonacci! Herz-Fischler suspects that Leonardus Pisanus may be a genuine fifteenth-century mathematician, and nothing to do with Leonardo da Pisa.

51 He lists the first four perfect numbers, 6, 28, 496 and 8128.

52 Conrad Matthaei, *Kurtzer, doch ausfuhrlicher Bericht von der Modis Musicis* (Königsberg, 1652), p. 15 reads: 'Wer hievon mit mehren wil berichtet seyn besehe die 36. Prop: des 9. Buchs Euclids'. I am grateful to Anna-Lena Holm, Senior Librarian, Rare Collections Statens musikbibliotek, Stockholm Sweden for making this information available to me at short notice.

intriguing to find a mathematical term described in a musical dictionary, and for the purposes of my argument, it is significant to find a reference to a music treatise followed by a reference to Euclid's original description of the phenomenon. Let us imagine that the geometrical term DEMR, or in Zedler's terms 'Sectio Divina', had been important for composers in 1732. Would it not be reasonable to expect Walther to define it as he did 'Numerus Perfectus'? Had DEMR been important for composers in 1732, I would expect Walther to have a) given a technical description, b) referred the reader to a musical treatise where it was set in the context of music theory or practice, and c) referred the reader to Euclid.[53] But he did not.

In summary, maths history has shown the theoretical possibility that European composers may have known a numerical equivalent of 'Sectio divina' after 1600. However, the published silence, or lack of coverage, in Zedler and Walther suggest either that there was no interest in the phenomenon, or that the new mathematical discoveries had not reached Leipzig. DEMR was probably not as divine a proportion to composers in Bach's time as some musicologists would have us think.

Conclusion[54]

There is a methodological weakness in much musicological literature on the golden section, whereby the author fails to appreciate a distinction between the composer's conscious compositional effort and the properties of the composition. The golden section is a naturally occurring phenomenon. Thus finding it in a work of art does not indicate that the composer deliberately put it there. Newman Powell strongly implies that Machaut and Dufay intended to incorporate the golden section in some of their compositions. Allan Atlas goes further and bases an interpretation on this implication.

> The significance seems clear: Dufay marks the structural point of the piece as a whole with the melodic and articulatory high point of the very phrase that contains in a nutshell the symbolic meaning of the work.[55]

This methodological weakness could be overcome were the author prepared to raise questions, instead of making statements. Atlas' discoveries would have been of more use had he asked rather than stated. For example, is there a high

53 It is unclear what theoretical and practical use Numerus Perfectus was to the musician in Matthei and Walther's time.

54 I discovered recently that I had been working on this problem at the same time as the Dutch academic Albert van der Schoot. His book was published in Dutch in 1998. Neither of us knew about the other. We were both assessing the subject by examining the sources from first principles, studying the primary sources and critically assessing secondary sources. Our conclusions are virtually identical, and echo those of Wittkower and Herz-Fischler, although the researches of all four of us were differently motivated. Far from weakening the impact of research, the evidence from four independent witnesses surely strengthens the results.

55 Atlas, *Acta Musicologica,* 59, pp. 122-23.

point in the 271st minim, did Dufay consciously organise his composition to climax at the golden section, or is this yet another example of a naturally occurring, aesthetically-pleasing proportion, rather than 'Dufay marks the structural point of the piece as a whole'.

Yet to reformulate results within a stricter methodological framework would not solve the problem. As I have shown, Powell and Atlas base their argument on historical facts that are at best implausible and at worst wholly incorrect.

Powell's approach is subtle. He works by implication from a misinformed interpretation of Nichomachus, through Boethius and Fibonacci to an overwhelming demonstration of selected golden sections in works by Machaut and Dufay. His scientific and experimental approach is laudable: the academic pedigree impeccable. Having been read at a meeting of the American Musicological Society and subsequently published in the *Journal of Music Theory*, his article has become the reference point to which many scholars look for justification of the use of the golden section and Fibonacci numbers in music of this period.

There are many snares in the path of the musicologist interested in the golden section and its aesthetic qualities. Not least among these is the trap of depending upon what might prove to be an unreliable secondary source. In this article I have presented evidence from different sources and disciplines, each of which contributes to form a picture of the way the golden section and Fibonacci numbers might have been used by composers. The research field was vast. Checking every secondary and primary source was time-consuming and complex, and in the end, in spite of the wealth of historical principles, several imponderable issues remain.

a) If knowledge of DEMR was kept secret, we are unlikely to discover a published description.
b) We do not know where Fibonacci learnt his sequence, and if it was really he who 'invented' it. Tradition tells that he was a merchant who traded off the coast of Africa and in Egypt. I understand that natural rabbits (the creature he uses to illustrate his tale) do not breed at the rate he described. On the other hand, smaller rodents, such as desert rats living off the coast of Africa and in Egypt, do.
c) Did the sequence spread to Europe via a route that bypassed Pacioli in Rome?
d) In the absence of a numerical sequence, might composers have used compasses to find the divine proportion on the ground plan of their compositions?

But in spite of the imponderables, there are a number of indisputable historical principles that should guide the analyst:

i) c. 300 BCE Euclid had no numbers to express DEMR.
ii) c. 100 CE Nicomachus' tenth proportion was not the Fibonacci sequence.
iii) c. 500 CE Boethius did not describe the Fibonacci sequence.

iv) 1509 Pacioli had no numbers to express. He did not apply DEMR or the Fibonacci sequence to lettering, architecture or the human form. He used simple proportions.
v) 1571 Jacob and 1611 Kepler published numbers for DEMR.
vi) Eighteenth-century mathematicians experimented with the sequence as an expression of DEMR.
vii) 1835 Martin Ohm used the term 'golden section'.
viii) 1878 Lucas named the Fibonacci sequence.
ix) With the spread of golden numberism in the early twentieth century it becomes increasingly likely that artists decide to experiment with Fibonacci numbers as an expression of the golden section.
x) Any composer using the additive numerical sequence before 1857 had no idea it was first mentioned by Leonardo da Pisa (Fibonacci).
xi) Any composer using the additive numerical sequence before c. 1600 would have had no idea that it was an approximation of Euclid's DEMR or Pacioli's *De Divina Proportione*.
xii) There is no documentary evidence to suggest that composers between the period c. 1600 and c. 1830 had any interest in using Euclid's ratio or the additive sequence in order to emulate a 'divine proportion'.
xiii) Since golden numberism first appeared in the mid-nineteenth and developed in the early twentieth century, the silence on the subject in Zedler's *Lexikon,* written and published in Leipzig between 1732 and 1754, should be no surprise.

Golden numberism has thoroughly infected musicology. There is a cure.